General Relativity

Final Exam
4/11/2016

Please write your first and last name and your student number on the first page.

Problem 1

Consider the two dimensional metric

$$ds^2 = \frac{1}{(1-r^2)^2}(dr^2 + r^2 d\theta^2)$$

with $0 \leq r < 1$.

1. Write the components of the metric and compute the Christoffel symbols.
2. How many independent components does the Riemann tensor have in 2 dimensions? Using these symmetries, compute the Ricci scalar of this manifold.
3. Derive the geodesic equations. Find the solution to the geodesic equation connecting a point $r_0, \theta = 0$ to the point $r_0, \theta = \pi$ and compute the length of this geodesic as a function of r_0.

Problem 2

Consider the Schwarzchild metric

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

i) Find the lowest bound for the radius a **stable** circular orbit for a massive particle.

ii) Compute the period of the circular orbit at this lowest bound: a) according to an observer at infinity b) according to the proper time of an observer moving together with the orbiting particle.

iii) Calculate the same for the innermost unstable circular orbit.

Problem 3

In this problem we want to study how gravitational waves would behave in a 3-dimensional universe (as opposed to our 4-dimensional one). Consider the Einstein equations in 3 space-time dimensions, in the absence of matter.

i) Expand the metric around 3d flat Minkowski space, and derive the linearized Einstein equations.

ii) State what is the "gauge invariance" of the equations, and use it to simplify the linearized Einstein equations.

iii) By following a similar analysis as the one in 4d, determine what are the possible polarizations of gravitational waves in 3d (Hint: the result has some surprising aspects).